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Operators Arising from Representations of Nilpotent Lie Groups*

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The following two results are obtained for an irreducible multiplier representation T of a connected nilpotent Lie group. First, T_f is a Hilbert–Schmidt operator if f is square-integrable with compact support. Second, T_f is of trace class if f has derivatives with sufficiently many moments. An application is made of the latter result to show that T_f can be of trace class even when f is not continuous.

1. INTRODUCTION

Suppose G is a locally compact group and that T is an irreducible representation of G. Let us consider the operators $[T_f]$ defined by $\int f(x)T_x\, dx$ whenever this integral makes sense. It is a mathematically fascinating question as to how the operator–theoretic properties of the T_f's are related to the function–theoretic properties of the f's. For which functions f is the operator T_f a compact operator, Hilbert–Schmidt, of trace class, a projection, etc.? Not only is this relationship a kind of mathematical intrigue, but in fact such knowledge has been useful in classifying groups, analyzing representations, and computing quantities important to the theory, e.g., Plancherel measure. Indeed questions of this sort must be the first to be posed once the connection between representations of a group and representations of its group algebra has been noted.

If case T is unitary, it is called CCR if T_f is a compact operator whenever f is integrable. Since the mapping $f \to T_f$ is continuous with respect to the L^1 norm, we see that T is CCR whenever T_f is compact for every f in a dense subset of $L^1(G)$, e.g., continuous functions with compact support, differentiable functions, etc. Connected semisimple Lie groups, connected nilpotent Lie groups, and motion groups, (semidirect product of a vector group with a compact group), all have the property that each of their irreducible unitary representations is

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CCR, and this undoubtedly is one of the reasons these groups are so much better understood than, for instance, discrete groups and solvable groups.

Somewhat more subtle is the question of which functions f map to Hilbert–Schmidt operators. In [1] it is shown that this is the case for every L^1 function if and only if T is finite dimensional, assuming still that T is irreducible and unitary. On the other hand, Harish–Chandra [3] has shown that T_f is Hilbert–Schmidt when G is connected and semisimple, T is irreducible and unitary, and f is square-integrable with compact support. As a special case of the results in [7] we see that T_f is Hilbert–Schmidt when G is a motion group, T is irreducible and unitary, and f is continuous with compact support. It also appears that Schochetman's same proof extends to functions which are square-integrable with compact support. Some time ago the author observed that the same result is true when T is an irreducible type I multiplier representation of an abelian group. See Theorem 2.4 below. We shall also prove, in Theorem 2.3 below, that T_f is Hilbert–Schmidt whenever G is a connected nilpotent Lie group, T an irreducible multiplier representation of G, and f is square-integrable with compact support. Actually, even if T is a unitary representation, in the inductive step of the proof given here one is confronted with a multiplier representation so that it is simplest to state the result at the outset for multiplier representations. Of course a multiplier representation of a nilpotent group corresponds, in the usual way, to an ordinary representation of another nilpotent group, so that this apparently more general statement is really no improvement. In our study of the relation between functions and trace class operators (see Section 4), there does seem to be an essential difference between the unitary result and the multiplier result.

Although from the above discussion the square-integrable functions with compact support seem to form an appropriate class of functions relative to Hilbert–Schmidt operators, we mention two related facts. Theorem 4.2 of [5] shows that certain representations of Lie groups having "Large" compact subgroups map sufficiently smooth functions to Hilbert–Schmidt operators. Also, square-integrable representations map every L^2 function to a Hilbert–Schmidt operator. See, for example, [6].

More delicate yet are the operators of finite trace. The trace norm of a bounded operator is a bit tricky to compute or even to estimate, and it is not at all continuous with respect to any of the usual operator topologies. Which functions f map, under a representation T, to operators of finite trace (Trace class operators), is of great interest since if this class is plentiful then a linear functional $f \to sp(T_f)$ exists, and this functional often plays a role similar to that of the character of a finite group. One of the major results in semisimple theory is that T_f is of trace class whenever T is irreducible and unitary and f is infinitely differentiable with compact support. The same result is valid for irreducible unitary representations of connected nilpotent Lie groups. In fact somewhat more is true in that case.

1.1. THEOREM (Kirillov). *Let G be a connected nilpotent Lie group and T an irreducible unitary representation of G. Then there exists an element D of the left enveloping algebra $\mathscr{U}(G)$ of G so that the operator T_f is of trace class, and $\| T_f \|_{sp} \leqslant \| DF \|_1$, whenever f is sufficiently smooth so that $T_{Df} = T_D T_f$.*

(Here "sufficiently smooth" means that the differential operator D can be passed through the integral sign in $\int f(x) T_x \, dx$. This is justified, for example, if $D'f$ is integrable for sufficiently many elements D' of $\mathscr{U}(G)$.)

From this theorem we see that "compact support" is by no means a requirement in order that T_f be of trace class. Rather it is some kind of integrability condition on the derivatives of f that is essential. In Section 4 we shall prove a result for multiplier representations which is analagous to Theorem 1.1. As a consequence of that result we shall discover that the "differentiability" condition in Theorem 1.1 is also by no means necessary. In fact there exist discontinuous functions which nevertheless map, under an irreducible unitary representation, to operators of finite trace. This is a consequence of our theorem (Theorem 4.1), and the theorem itself appears more restrictive than the one above. What really is proved there is the following:

THEOREM. *Let T be an irreducible multiplier representation of a connected nilpotent Lie group G. Then there exist a finite number $D^1, \ldots, D^t$ of elements of $\mathscr{U}(G)$ and a corresponding number $p^1, \ldots, p^t$ of polynomial functions on G such that T_f is of trace class whenever $p^i D^i f$ is integrable for each i. Further,*

$$\| T_f \|_{sp} \leqslant \int_{i=1}^{t} \| p^i D^i f \|_1 .$$

To carry out our proof we must require, in addition to the integrability of certain derivatives of f, the existence of certain "Moments" of these derivatives. It is only after having proved this theorem that we find we can drop the differentiability condition in Theorem 1.1 as a necessary assumption.

Whether Theorem 1.1 holds as it stands for multiplier representations is still unclear. The author has tried hard to discover a counterexample without success. There are reasons why one would expect 1.1 not to hold for multiplier representations. In the first place, since the multiplier might not even be continuous let alone differentiable, there might not be any differentiable vectors at all and therefore no operator T_D. In the second place, even when the multiplier is analytic and all the operators T_D are densely defined, it is still not true in general that $T_D T_f = T_{Df}$. These two facts are central to Kirillov's proof. There is also some reason to feel that 1.1 would hold for multiplier representations. Indeed one should be able to construct a proof by passing up to the group extension defined by the multiplier, applying 1.1 to that group, and then, "projecting" back down. The proof given here is based on this idea, but the "projecting" back down is not so simple, and the polynomial coefficients seem inevitably to enter in. The following example may make the complexities of this situation more apparent.

1.2. EXAMPLE. Let g be an analytic, nonplolynomial, function of a real variable, and define a function b from R^2 into the circle group L by $b(q, p) = e^{ipg(q)}$ Define a multiplier δ on $R^2 \times R^2$ by $\delta((q_1, p_1), (q_2, p_2)) = e^{i(q_2 p_1)}b(q_1 p_1)b(q_2, p_2)/b(q_1 + q_2, p_1 + p_2)$. The essentially unique δ-representation T of R^2 is defined on $L^2(R)$ by $(T_{(q,p)}(\varphi), \varphi) = b(q, p) \int_R e^{iqm} \varphi(p + m) \bar{\varphi}(m)\, dm$. Let G_1 denote the group extension of R^2 by L defined by δ, and let T_1 be the unitary representation of G_1 defined by $T_{1_{(\lambda,g)}} = \lambda T_g$. Let $a(\lambda)$ be a smooth function on L such that $\int_L a(\lambda)\lambda\, d\lambda = 1$, and for any function f 9n R^2 put $f_1(\lambda, g) = a(\lambda) f(g)$. Then the operators T_f and $T_{1_{(f_1)}}$ are equal.

Now according to Theorem 1.1, the operator T_f will be of trace class providing sufficiently many derivatives of f_1 are integrable over G_1. However this integrability of derivatives of f_1 can translate into something quite different about f. For example, differentiating along the one-parameter subgroup $(1, 0, t)$ in G_1 we have:

$$\begin{aligned}
&(d/dt)\, [f_1((\lambda, q, p)(1, 0, t))](0) \\
&= (d/dt)\, [f_1(\lambda\delta((q, p), (0, t)), q, (p + t))](0) \\
&= (d/dt)\, [f_1(\lambda e^{(ipg(q)+tg(0)-(p+t)g(q))}, q, (p + t))](0) \\
&= (d/dt)\, [a(\lambda e^{it(g(0)-g(q))}) f(q, (p + t))](0) \\
&= a'(\lambda)\, \lambda i(g(0) - g(q)) f(q, p) + a(\lambda)\, d_2 f(q, p).
\end{aligned}$$

Presumably higher order derivatives of f_1 would lead to even more complicated differential operators acting on f. It seems clear that we would have to consider differential operators with analytic, nonpolynomial, coefficients, and that we would be forced to assume the integrability of such derivatives of f. Therefore the fact that we can get away with polynomial coefficients is already interesting.

If we integrate out the λ, in order to project back down to G, it appears that the only way we can bound the trace norm of T_f, that is the trace norm of $T_{1_{(f_1)}}$, is with a sum of L^1 norms of moments of derivatives of f.

The integrability of derivatives of a function, the existence of moments of those derivatives, and for that matter the very definition of "polynomial function" on a group, is crucially dependent on the coordinate system being used. In Section 3 we shall investigate this question in some detail.

It is possible to prove Theorem 4.1 so that the operators $D^1,..., D^t$ and the polynomial functions $p^1,..., p^t$ are independent of the multiplier, i.e., the same operators and polynomials work for cohomologous representations. The proof presented here does not show this, but the other argument, although elementary in some sense, is long and does not seem to merit the space.

2. Hilbert–Schmidt Operators

The purpose of this section is to prove that T_f is a Hilbert–Schmidt operator whenever T is an irreducible unitary representation of a connected nilpotent Lie group and f is a square-integrable function with compact support. The proof is by induction on the dimension of the group, and in the inductive step we are confronted with a multiplier representation instead of an ordinary one and an apparently unavoidable "unipotent" automorphism. We must state our theorem then in a somewhat more general-souding form. The technicalities of the proof involve the structure of nilpotent groups, in particular the constructions introduced by Kirillov in [4] for the case of a one-dimensional center. We recall this structure below. Throughout the section, π will denote the projection of one group onto a quotient group. The context should make clear which groups are involved.

2.1. Let G be a simply connected nilpotent Lie group of dimension n, and let $\mathscr{G}$ denote its Lie algebra. Denote by Z the center of G and by $\mathscr{Z}$ its Lie algebra.

2.1.1. There exists a measure-preserving cross-section p of G/Z into G, and for any such cross-section we have

$$\int_G f(g)\, dg = \int_{G/Z} \int_Z f(zp(y))\, dz\, dy$$

for properly normalized Haar measures. (Indeed p can be taken to be a diffeomorphism.)

2.1.2. Now let Z be one-dimensional. Then, according to [4], there exist nonzero elements x, y, and z in $\mathscr{G}$ such that z generates Z, x does not belong to the commutator subalgebra $\mathscr{G}_1$ of $\mathscr{G}$, and the bracket $[x, y]$ of x and y is z. Let $\mathscr{G}_0$ be the annihilator of y in $\mathscr{G}$. Then $\mathscr{G}_0$ is an ideal in $\mathscr{G}$ of dimension $n-1$. It is important to note that whenever elements x, y, z exist satisfying the above bracket, and for which the annihilator of y is of codimension 1, the following constructions are valid. We write Y, X, G_0 for the closed subgroups of G having Lie algebras $[y]$, $[x]$, and $\mathscr{G}_0$, respectively. The product YZ is a subgroup of G which is contained in the center of G_0, and we denote by K the quotient group G_0/YZ. It will be to K that we apply our inductive hypotheses.

2.1.3. Let G be a simply connected nilpotent Lie group, and let $x_1, \ldots, x_n$ be a basis of its Lie algebra. Then the mapping $(t_1, \ldots, t_n) \to \prod_{i=1}^{n} \exp(t_i x_i)$ is a measure-preserving difeomorphism of R^n onto G.

2.1.4. Let G be as in 2.1.3. Let the basis $[x_1]$ have the properties that $x_1 = z$, $x_2 = y$, $x_n = x$, and the span of $x_1, \ldots, x_{n-1}$ is $\mathscr{G}_0$. Then:

(i) The mapping $\prod_{i=3}^{n-1} \exp(t_i x_i) \to \prod_{i=3}^{n-1} \exp(t_i\, d\pi(x_i))$ is a difeomorphism whose inverse p is a measure-preserving cross-section of K into G.

(ii) If we write, as we can by 2.1.3 and (i) above, elements g of G uniquely as $g = \exp(tz)\exp(sy)\,p(k)\exp(qx)$, or in shorthand $g = tz \cdot sy \cdot p(k) \cdot qx$, then for any integrable function f on G we have:

$$\int_G f(g)\,dg = \int_R \int_K \int_R \int_R f(tz \cdot sy \cdot p(k) \cdot qx)\,dt\,ds\,dk\,dq.$$

2.1.5. Let G be as in 2.1.4 and let χ be the character on Z defined by $\chi(tz) = e^{it}$. Let T_1 be an irreducible unitary representation of G whose restriction to Z is a multiple of χ. Then, according to [4, Section IV], there exists and irreducible unitary representation S of G_0 such that:

(i) S is trivial on Y.

(ii) S restricts to a multiple of χ on Z.

(iii) T_1 is equivalent to U^S.

The Hilbert space of T_1 is then isomorphic with the Hilbert space of all square-integrable functions on G/G_0, which we may identify with X, into the space of S. By 2.1.3 we can write each element g in G as $g = g_0 \cdot qx$, and if v is an element of the space of T_1 we have:

$$[T_{1_g}(v)](r) = [U_g{}^S(v)](r) = [U^S_{(g_0 \cdot qx)}(v)](r) = S_{(-rx \cdot g_0 \cdot rx)}(v(r-q)).$$

We denote by S' the unique irreducible multiplier representation of K for which

$$S_{(tz \cdot sy \cdot p(k))} = e^{it} S_k'.$$

This completes the technical constructions necessary to our proof.

2.2. Definition. An automorphism w of a connected nilpotent Lie group is called *unipotent* if it is the identity modulo some descending series of connected normal subgroups. Such an automorphism leaves some nontrivial subgroup of the center pointwise invariant. A unipotent automorphism is measure-preserving. An example of a unipotent automorphism, indeed the one with which we must contend, is when G is a proper normal subgroup of another nilpotent group, and w is conjugation by an element not in G.

2.3. Theorem. *Let G be a connected nilpotent Lie group of dimension n, let T be an irreducible multiplier representation of G, and let C be a compact subset of G. Then*:

(i) *If f is square-integrable with support in C, then T_f is a Hilbert–Schmidt operator.*

(ii) *There exists a constant c, depending only on T and C, such that if f has support in C, then $\| T_f \|_{HS}^2 \leqslant c \| f \|_2^2$.*

(iii) *In fact, there exists a constant c, depending only on T and C, such that if f has support in C and w is any unipotent automorphism of G, then* $\| T_{(f\cdot w)} \|_{HS}^2 \leq c \| f \|_2^2$.

Proof. Clearly (iii) implies (ii) implies (i), so that we need only verify (iii). As a matter of fact (i) does not imply (iii) since the support of $f \cdot w$ changes with w.

If G is abelian and the multiplier is trivial, then T is one-dimensional, and the theorem follows by taking c equal to the measure of C. This takes care of the cases $n = 0$ and $n = 1$. Assume then that $n \geq 2$.

There exists a multiplication on the set G_1 of all pairs (t, g), for t real and g in G, which makes G_1 into a connected nilpotent Lie group, and for which the mapping T_1 which sends the pair (t, g) to $e^{it}T_g$ is an irreducible unitary representation of G_1. Indeed G_1 is a covering group for the usual group extension of G by the circle group associated with the multiplier. Let $a(t)$ be an integrable function on R for which $\int_R a(t)e^{it}\, dt = 1$.

Let w be a unipotent automorphism of G. We treat first the case when the center of G_1 contains a subgroup Z for which Z/R is a subgroup N of positive dimension which is pointwise invariant under w. Since the abelian Lie group Z is isomorphic with the product $R \times N$, there exists a measure-preserving cross-section p' of G into G_1 which is multiplicative on N. For any function f on G, put $f_1((t, e)\, p'(g)) = a(t) f(g)$. Then the operators T_f and $T_{1_{(f_1)}}$ are equal. Let χ denote the character of Z for which $T_{1|_Z}$ is a multiple of χ. Let p'' be a measure-preserving cross-section of G/N into G, and let T_2 be the unique multiplier representation of G/N for which

$$T_{1_{((t,e)p'(np''(y)))}} = \chi((t, e)\, p'(n))\, T_{2_y}\,.$$

Now if f has support in C and then

$$T_{(f\cdot w)} = T_{1_{((f\cdot w)_1)}} = \int_{G_1} (f \cdot w)_1\, (g_1)\, T_{1_{(g_1)}}\, dg_1$$

$$= \int_{G/N} \int_N \int_R (f \cdot w)_1\, ((t, e)\, p'(np''(y)))\, \chi((t, e)\, p'(n)\, dt\, dn\, T_{2_y}\, dy$$

$$= \int_{G/N} \int_N \int_R a(t)\, e^{it}\, dt (f \cdot w)(np''(y))\, \chi(p'(n))\, dn\, T_{2_y}\, dy$$

$$= \int_{G/N} \int_N f(nw(p''(y)))\, \chi(p'(n))\, dn\, T_{2_y}\, dy$$

(because w leaves N pointwise invariant)

$$= \int_{G/N} \int_N f(nn(w, y)\, p''(w_2(y)))\, \chi(p'(n))\, dn\, T_{2_y}\, dy,$$

where w_2 is the automorphism of G/N induced by w. Of course w_2 is unipotent. Therefore

$$T_{(f\cdot w)} = \int_{G/N} \int_N f(np''(w_2(y)))\, \chi(p'(n(n(w, y))^{-1}))\, dn\, T_{2_y}\, dy$$
$$= \int_{G/N} (f_2 \cdot w_2)(y)\, T_{2_y}\, dy,$$

where $f_2(y) = \int_N f(np''(y))\, \chi(p'(n))\, dn\, \chi(p'(n(w, w_2^{-1}(y))))$. Therefore $T_{(f\cdot w)} = T_{2_{(f_2\cdot w_2)}}$.

Now the support of f_2 is compact. Indeed it is contained in the set $\pi(C)$. Since the dimension of $G/N < n$, we have from the inductive hypothesis that there exists a constant c_2, depending only on T_2 and $\pi(C)$, i.e., depending only on T and C, such that

$$\| T_{(f\cdot w)} \|_{HS}^2 = \| T_{2_{(f_2\cdot w_2)}} \|_{HS}^2$$
$$\leqslant \int_{G/N} |f_2(y)|^2\, dy\, c_2$$
$$= c_2 \int_{G/N} \left| \int_N f(np''(y))\, \chi(p'(n))\, dn \right|^2 dy$$
$$\leqslant c_2 \int_{G/N} \left[\int_N |f(np''(y))|\, dn \right]^2 dy$$
$$\leqslant c_1 c_2 \int_{G/N} \int_N |f(np''(y))|^2\, dn\, dy$$
$$= c_1 c_2 \int_G |f(g)|^2\, dg,$$

where c_1 is the supremum, over all y in G/N, of the measure of the set of all n in N such that $np''(y)$ belongs to C. This supremum clearly is bounded by the diameter of the compact set $\pi(C)$ and so depends only on C. This establishes (iii) of the theorem in this case.

Assume next that no such subgroup Z of the center of G_1 exists. Let y be a vector in the Lie algebra of G_1 such that $y_0 = d\pi(y)$ generates a one-parameter subgroup of the center of G which is pointwise invariant under w. Because we are in this second case, y does not belong to the center of G_1, and there must exist an x in the Lie algebra of G_1 such that $[x, y] = z$ a vector which generates the subgroup R of G_1. The annihilator $\mathscr{G}_0$ of y is of dimension n, so that we may presume all of the constructions introducted in 2.1. Let $x_0 = d\pi(x)$. Finally let p''' be a measure-preserving cross-section of K into G. We make these definitions in order to employ a different correspondence between functions on G and functions on G_1. If f is a function on G, put

$$f_3(tz \cdot sy \cdot p(k) \cdot qx) = a(t)\, f(sy_0 \cdot p'''(k) \cdot qx_0).$$

The operators T_f and $T_{1_{(f_3)}}$ are equal. (In this case G itself is simply connected, and all the integration formulas of 2.1.4 hold.)

Let $[\varphi_i]$ be an orthonormal basis for $L^2(R)$ and let $[\psi_j]$ be an orthonormal basis for the space $H(S)$ of S. Define an orthonormal basis $[v_{ij}]$ for the space of T_1 by $v_{ij}(r) = \varphi_i(r)\psi_j$. Now if f has support in C

$$\begin{aligned}
\| T_{(f\cdot w)} \|_{HS}^2 &= \| T_{1_{((f\cdot w)_3)}} \|_{HS}^2 \\
&= \sum_i \sum_{i'} \sum_j \sum_{j'} |(T_{1_{((f\cdot w)_3)}}(v_{ij}), v_{i'j'})|^2 \\
&= \sum_i \sum_{i'} \sum_j \sum_{j'} \left| \int_R ([T_{1_{((f\cdot w)_3)}}(v_{ij})](r)), v_{i'j'}(r))_{H(S)}\, dr \right|^2 \\
&= \sum_i \sum_{i'} \sum_j \sum_{j'} \left| \int_R \int_{G_1} ((f\cdot w)_3\,(g_1)([U_{g_1}^S(v_{ij})](r), v_{i'j'}(r))_{H(S)}\, dg_1\, dr \right|^2 \\
&= \sum_i \sum_{i'} \sum_j \sum_{j'} \left| \int_R \int_R \int_{G_0} (f\cdot w)_3\,(g_0\cdot qx) \right. \\
&\qquad \left. \times ([S_{(-rx\cdot g_0\cdot rx)}(v_{ij}(r-q))], v_{i'j'}(r))_{H(WS)}\, dg_0\, dq\, dr \right|^2 \\
&= \sum_i \sum_{i'} \sum_j \sum_{j'} \left| \int_R \int_R \int_{G_0} (f\cdot w)_3\,(g_0\cdot qx) \right. \\
&\qquad \left. \times (S_{(-rx\cdot g_0\cdot rx)}(\psi_j), \psi_{j'})_{H(S)}\, \varphi_i(r-q)\, \bar{\varphi}_{i'}(r)\, dg_0\, dq\, dr \right|^2 \\
&= \sum_j \sum_{j'} \sum_i \sum_{i'} \left| \int_R \int_{G_0} \int_R (f\cdot w)_3\,(g_0\cdot(r-q)x) \right. \\
&\qquad \left. \times (S_{(-rx\cdot g_0\cdot rx)}(\psi_j), \psi_{j'})_{H(S)}\, \varphi_i(q)\, \bar{\varphi}_{i'}(r)\, dq\, dg_0\, dr \right|^2 \\
&= \sum_j \sum_{j'} \sum_i \sum_{i'} \left| \int_R \int_R V^{jj'}(r,q)\, \varphi_i(q)\, \bar{\varphi}_{i'}(r)\, dq\, dr \right|^2,
\end{aligned}$$

where $V^{jj'}(r, q)$ is the kernel

$$\int_{G_0} (f\cdot w)_3\,(g_0\cdot(r-q)x)(S_{(-rx\cdot g_0\cdot rx)}(\psi_j), \psi_{j'})_{H(S)}\, dg_0 .$$

Hence

$$\begin{aligned}
\| T_{(f\cdot w)} \|_{HS}^2 &= \sum_j \sum_{j'} \int_R \int_R | V^{jj'}(r,q)|^2\, dq\, dr \\
&= \sum_j \sum_{j'} \int_R \int_R \left| \int_{G_0} (f\cdot w)_3\,(g_0\cdot qx) \right. \\
&\qquad \left. \times (S_{(-rx\cdot g_0\cdot rx)}(\psi_j), \psi_{j'})_{H(S)}\, dg_0 \right|^2 dq\, dr
\end{aligned}$$

$$= \sum_j \sum_{j'} \int_R \int_R \left| \int_K \int_R \int_R (f \cdot w)_3 \, (tz \cdot sy \cdot p(k) \cdot qx) \right.$$

$$\left. \times \, (S_{(-rx \cdot tz \cdot sy \cdot p(k) \cdot rx)}(\psi_j), \psi_{j'})_{H(S)} \, dt \, ds \, dk \right|^2 dq \, dr$$

$$= \sum_j \sum_{j'} \int_R \int_R \left| \int_K \int_R (f \cdot w)(sy_0 \cdot p'''(k) \cdot qx_0) \, e^{-isr} \, ds \right.$$

$$\left. \times \, (S_{(-rx \cdot p(k) \cdot rx)}(\psi_j), \psi_{j'})_{H(S)} \, dk \right|^2 dq \, dr$$

$$= \sum_j \sum_{j'} \int_R \int_R \left| \int_K \int_R f(sy_0 \cdot w(p'''(k)) \cdot w(qx_0)) \, e^{-isr} \, ds \right.$$

$$\left. \times \, (S_{(t(r,k)z \cdot s(r,k)y \cdot p(w_r(k)))}(\psi_j), \psi_{j'})_{H(S)} \, dk \right|^2 dq \, dr,$$

where w_r is the unipotent automorphism of K induced by conjugation by rx, and where $t(r, k)$ and $s(r, k)$ are real numbers. Hence

$$\| T_{(f \cdot w)} \|_{HS}^2 = \sum_j \sum_{j'} \int_R \int_R \left| \int_K \int_R f(sy_0 \cdot s(w, k) \, y_0 \cdot p'''(w'(k)) \cdot w(qx_0)) \right.$$

$$\left. \times \, e^{-isr} \, ds \, e^{it(r,k)} (S'_{(w_r(k))}(\psi_j), \psi_{j'})_{H(S)} \, dk \right|^2 dq \, dr$$

$$= \sum_j \sum_{j'} \int_R \int_R \left| \int_K \int_R f(sy_0 \cdot p'''(w'(k)) \cdot w(qx_0)) \, e^{-isr} \, ds \right.$$

$$\left. \times \, m_{(r,w)}(k) (S'_{(w_r(k))}(\psi_j), \psi_{j'})_{H(S)} \, dk \right|^2 dq \, dr,$$

where $m_{(r,w)}(k) = e^{it(r,k)} e^{irs(w,k)}$. Recalling again that unipotent automorphisms are measure-preserving, we have finally that

$$\| T_{(f \cdot w)} \|_{HS}^2 = \sum_j \sum_{j'} \int_R \int_R \left| \int_K (f^{(r,q,w)} \cdot w' \cdot (w_r)^{-1})(k) \right.$$

$$\left. \times \, (S_k'(\psi_j), \psi_{j'})_{H(S)} \, dk \right|^2 dq \, dr$$

$$= \int_R \int_R \| S'(f^{(r,q,w)} \cdot w' \cdot (w_r)^{-1}) \|_{HS}^2 \, dr \, dq,$$

where

$$f^{(r,q,w)}(k) = \int_R f(sy_0 \cdot p'''(k) \cdot w(qx_0)) \, e^{-isr} \, ds \, m_{(r,w)}(w'^{-1}(k)).$$

Now the support of $f^{(r,q,w)}$ lies in the compact set $\pi(C)$ which depends only on C, and so by the inductive hypotheses there exists a constant c, depending only on $\pi(C)$ and S' and consequently only on C and T, such that

$$\|T_{(f\cdot w)}\|_{HS}^2 \leqslant c\int_R\int_R\int_K |f^{(r,q,w)}(k)|^2\,dk\,dr\,dq$$
$$= c\int_R\int_K\int_R \left|\int_R f(sy_0\cdot p'''(k)\cdot w(qx_0))\,e^{-isr}\,ds\right|^2 dr\,dk\,dq$$
$$= 2\pi c\int_R\int_K\int_R |f(sy_0\cdot p'''(k)\cdot w(qx_0))|^2\,ds\,dk\,dq$$
$$= 2\pi c\int_R\int_K\int_R |f(sy_0\cdot p'''(k)\cdot q\,dw(x_0))|^2\,ds\,dk\,dq$$
$$= 2\pi c\int_G |f(g)|^2\,dg$$

by 2.1.3. This completes the proof of the theorem.

Essentially as a corollary to this theorem we have the following:

2.4. THEOREM. *Let G be an abelian locally compact group, δ a type I multiplier on $G\times G$, T an irreducible δ-representation of G, and C a compact subset of G. Then there exists a constant c, depending only on T and C, such that if f is square-integrable with support in C, then the operator T_f is Hilbert–Schmidt, and*

$$\|T_f\|_{HS}^2 \leqslant c\|f\|_2^2.$$

Proof. As usual in this subject, one may immediately reduce to the case when δ is "totally skew"; see [2]. Then G is a direct product $G_1\times G_2$ where G_1 is R^{2n} and G_2 is a finite group. Further, T is the outer Kronecker product $T^1\times T^2$ of irreducible multiplier representations T^1 of G_1 and T^2 of G_2, and T^2 is finite dimensional. Now if f has support in C, then T_f

$$= \sum_{g_2\in G_2}\int_{G_1} f(g_1+g_2)\,T^1_{g_1}\,dg_1\,T^2_{g_2}/O(G_2),$$

where $O(G_2)$ is the order of that finite group. Hence T_f is a finite sum of operators which are outer products of finite-dimensional operators and Hilbert–Schmidt operators (by the last theorem), and so T_f itself is Hilbert–Schmidt. Q.E.D.

Theorem 2.3 does imply that T_f is Hilbert–Schmidt whenever f is continuous with compact support, so that, in the terminology of [7], connected nilpotent Lie groups are H.S.-groups.

3. INTEGRABILITY OF DERIVATIVES

The question of whether a function is differentiable at a point on a manifold is of course independent of the coordinate system at the point. Whether a function is integrable on a Lie group is also independent of any coordinate system, the Haar measure being determined by the underlying topological group. However, if one wishes to investigate the global integrability of a derivative of a function on a Lie group, then the coordinate system becomes a key factor. More to the point, it is what we mean by a "Global Derivative" of f that depends on the coordinate system. Consider the following two examples.

3.1. EXAMPLE. Let G be the group of all triples (t, q, p) of real numbers with multiplication defined by $(t_1, q_1, p_1)(t_2, q_2, p_2) = (t_1 + t_2 + q_2p_1, q_1 + q_2, p_1 + p_2)$. This is a Lie group, and it is in fact the Heisenberg group. Its underlying manifold is obviously R^3, and one would assume that the first order derivatives of a function f would be the functions $(d/dt)f$, $(d/dq)f$, and $(d/dp)f$. These certainly are the simplest differential operators to apply to f in order to determine if it is differentiable at a point.

There are, however, some other first order differential operators which are in fact more intrinsic to the group itself. These are the differential operators obtained from the left invariant vector fields on G, i.e., by differentiating along one-parameter subgroups of G. For this group, these operators will be linear combinations of the following three. Differentiating along $(t, 0, 0)$, we obtain d/dt. Along the subgroup $(0, q, 0)$ we obtain the operator $d/dq + pd/dt$. And along $(0, 0, p)$, we obtain d/dp.

The point is that if we ask whether the first order derivatives of a function f are integrable, then we must be sure to specify which first order derivatives we mean. In this example the integrability of the first order derivatives arising from the vector fields would imply, in addition to the integrability of certain ordinary derivatives of f, the existence of some kind of "moment" with respect to the variable p.

3.2. EXAMPLE. The next group is nothing but the simply connected covering group of the group in Example 1.2. Thus let G be the group of triples (t, q, p) of real numbers with multiplication defined by

$$(t_1, q_1, p_1)(t_2, q_2, p_2) = (t_1 + t_2 + q_2p_1 + p_1g(q_1) + p_2g(q_2) - (p_1 + p_2)\, g(q_1 + q_2), q_1 + q_2, p_1 + p_2),$$

where g is an analytic but nonpolynomial function of a real variable. This is again a Lie group, and it is in fact isomorphic to the previous example. The underlying manifold is again obviously R^3, and again d/dt, d/dq, and d/dp seem to be the appropriate first order differential operators. This time, however, differentiating along the subgroup $(0, 0, p)$ we obtain the operator $d/dp + d/dt\, [g(0) - g(q)]$. Requiring that a function be integrable when acted upon by this operator is a much more stringent restriction than the existence of a moment. It should be clear too that other isomorphic copies of the Heisenberg group would produce first order operators with discontinuous coefficients. The picture is muddy, and we shall try to clear up at least a part of it.

Throughout the rest of this section we shall be discussing a connected nilpotent Lie group G. The results stated here may be well known to some experts, but we present them because the geometric handle obtained is crucial to the proof of Theorem 4.1. We give no proofs in this section, the arguments being nontrivial but routine consequences of nilpotency.

Let G be a connected nilpotent Lie group of dimension N, and let K be a maximal torus in G. Then it is known that K is a maximum torus and that it is contained in the center of G.

3.3. THEOREM. *Let $[x_i]$ be a basis for the Lie algebra $\mathscr{G}$ of G with the following two properties*:

(i) $x_1, \ldots, x_J$ *span the Lie algebra $\mathscr{K}$ of K, and K is the direct product of the closed one-parameter subgroups* $\exp(tx_i)$, *for* $1 \leqslant i \leqslant J$.

(ii) $[x_i]$ *is consistent with the central descending series, i.e., the Lie bracket of x_i with the span of $x_1, \ldots, x_{i-1}$ is contained in the span of $x_1, \ldots, x_{i-2}$*.

Then:

(A) Each element g of G can be written uniquely as $g = \prod_{i=1}^{n} \exp(t_i x_i)$, where $-l_i \leqslant t_i < l_1$ for $1 \leqslant i \leqslant J$ and t_i real for $i > J$. The mapping which sends $\prod_{i=1}^{n} \exp(t_i x_i)$ to the n-tuple $(\lambda_1, \ldots, \lambda_J, t_{J+1}, \ldots, t_n)$, $(\lambda_I = e^{i(\pi t_i/l_i)})$, thus defined is a diffeomorphism of G onto $L^J \times R^{n-J}$. (L denotes the circle group.) We denote this global coordinatization of G by $\varphi_{[x_i]}$.

(B) If $[y_i]$ is another basis of $\mathscr{G}$ which satisfies i and ii, then $\varphi_{[y_i]} \cdot [\varphi_{[x_i]}]^{-1}$ has polynomial component functions and a constant Jacobian determinant.

(C) $\prod_{i=1}^{n} \exp(t_i x_i) \prod_{j=1}^{n} \exp(t_j' x_j) = \prod_{k=1}^{n} \exp(p_k x_k)$, where $p_k = t_k + t_k' + p_k'$, and p_k' is a polynomial in the variables $(t_{k+1}, \ldots, t_n, t_{k+1}', \ldots, t_n')$.

(D) If f is an integrable function on G, then

$$\int_G f(g)\, dg = \int_{(-l_1)}^{(l_1)} \cdots \int_{(-l_J)}^{(l_J)} \int_R \cdots \int_R f\left(\prod_{i=1}^{n} \exp(t_i x_i)\, dt_1 \cdots dt_n\right),$$

where dt_i represents properly normalized Haar measure on the one-parameter group $\exp(tx_i)$.

3.4. DEFINITION. Any basis $[x_i]$ of the Lie algebra $\mathscr{G}$ of G which satisfies conditions (i) and (ii) of the above theorem is called an *exponential coordinate system* for G. A function p on G is a *polynomial function* if it is a polynomial when expressed in an exponential coordinate system. By part (B) of the above theorem, it follows that p is a polynomial in any other exponential coordinate system, although its degree could change.

3.5. THEOREM. *Denote by $\mathscr{U}(G)$ the left enveloping algebra of G, i.e., the algebra (associative) generated by the linear differential operators arising from the left invariant vector fields on G. Let $[x_i]$ be an exponential coordinate system for G, and denote by $\mathscr{E}(G)$ the algebra of linear differential operators generated by the first order derivatives d/dx_i with respect to this coordinate system. Then:*

(i) *$\mathscr{E}(G)$ is always abelian, but $\mathscr{U}(G)$ is abelian if and only if G is commutative*

(ii) *For each element D in $\mathscr{U}(G)$ there exist elements $E^1,..., E^r$ in $\mathscr{E}(G)$ and polynomial functions $p^1,..., p^r$ on G such that $D = \sum_{i=1}^{r} p^i E^i$.*

(iii) *For each element E in $\mathscr{E}(G)$ there exist elements $D^1,..., D^s$ in $\mathscr{U}(G)$ and polynomial functions $p^1,..., p^s$ on G such that $E = \sum_{i=1}^{s} p^i D^i$.*

(iv) *If $\mathscr{U}_p(G)$ is the algebra generated by differential operators of the form pD for p a polynomial function on G and D an element of $\mathscr{U}(G)$, and if $\mathscr{E}_p(G)$ is the algebra generated by the operators of the form pE for p a polynomial function and E an element of $\mathscr{E}(G)$, then $\mathscr{U}_p(G) = \mathscr{E}_p(G)$.*

This is definitely a theorem about nilpotent groups. We have that $\mathscr{U}(G)$ is generated by the vector fields $[x_i]$, and $\mathscr{E}(G)$ is generated by the operators d/dx_i. Now (ii) and (iii) follow by induction together with Theorem 3.3, part (C). Part (iv) follows from (ii) and (iii), and (i) is a fact true in great generality.

Let us have another look at the examples given in the beginning of this section. In Example 3.1 the evident coordinatization of G by R^3 is an exponential coordinatization, i.e., $(t, q, p) = (t, 0, 0)(0, q, 0)(0, 0, p)$, and the generators for these three one-parameter subgroups form an exponential coordinate system. From our calculations we see that the differential operators arising from thevector fields are, with respect to the exponential coordinate system, linear differential operators with polynomial coefficients, and we can easily see how to recover the first order operators d/dt, d/dq, and d/dp from the vector fields and polynomial coefficients.

In Example 3.2 the evident coordinatization of G by R^3 is not an exponential coordinatization, for $(t, 0, 0)(0, q, 0)(0, 0, p) = (t + p(g(0) - g(q)), q, p)$ and not (t, q, p) as desired. Our calculations here showed that an element of $\mathscr{U}(G)$, i.e., a differential operator arising from a left invariant vector field, could not be expressed in terms of these coordinates merely with polynomial coefficients. The point seems to be that this "evident" coordinate system is simply the wrong one for this group. Unfortunately the novice may not notice this.

How then shall we define when a function f has integrable derivatives. Clearly the most elegant definition would be to require that Df be integrable for D in $\mathscr{U}(G)$. Checking this condition in practice involves discovering the algebra $\mathscr{U}(G)$ which is far more likely to be elusive than is the algebra $\mathscr{E}(G)$ coming from an exponential coordinate system. However the algebra $\mathscr{E}(G)$ has very little to do with the group structure, and from different exponential coordinate systems we would derive different algebras. (The integrability of a function is independent of which exponential coordinate system is used because of the constant Jacobian determinant. See Theorem 3.3, part (B)).

For our purposes the dilemma has a happy ending.

In the theorem in the next section we find that it is elements of $\mathscr{U}_p(G)$ that must be considered. Once one is committed to working with operators having polynomial coefficients, the algebra $\mathscr{E}_p(G)$, which after all is $\mathscr{U}_p(G)$, is the simpler algebra.

4. Trace Class Operators

The purpose of this section is to prove that the operator T_f is of trace class whenever T is an irreducible multiplier representation of a connected nilpotent Lie group and f is a function whose derivatives have sufficiently many moments. Unlike the result in Section 2, this theorem is already known for the unitary case; see Theorem 1.1. The multiplier result is apparently different in the sense that no single operator exists, and the operators which do ensure that T_f is of trace class are operators with polynomial coefficients.

4.1. Theorem. *Let T be an irreducible multiplier representation of a connected nilpotent Lie group G of dimension n. Then there exist elements $D^1,\dots, D^t$ in the left enveloping algebra $\mathscr{U}(G)$ of G and polynomial functions $p^1,\dots, p^t$ on G such that the operator T_f is of trace class whenever p^iD^if is integrable for each i. Further, $\|T_f\|_{s,p} \leq \sum_{i=1}^t \|p^iD^if\|_1$.*

Proof. Let the dimension of the maximum torus K in G be equal to J. Let δ be the multiplier associated with the representation T, and let G_1 be the group extension of G by the circle group L defined by δ. Define T_1 on G_1 by $T_{1_{(\lambda,y)}} = \lambda T_y$, and let $a(\lambda)$ be an infinitely differentiable function on L for which $\int_L a(\lambda)\lambda\, d\lambda = 1$. Let $[x_1,\dots, x_{n+1}]$ be an exponential coordinate system for G_1 (see Definition 3.4.) Assume without loss of geneality that x_1 spans the Lie algebra of L. If π denotes projection of G_1 onto G, then the elements $[d\pi(x_2,\dots, d\pi(x_{n+1})]$ form an exponential coordinate system for G. For any function f on G, put

$$f_1\left(\prod_{i=1}^{n+1} \exp(t_ix_i)\right) = a(\exp(t_1x_1)\, f\left(\prod_{i=2}^{n+1} \exp(t_i\, d\pi(x_i))\right).$$

Then T_f and $T_{1_{(f_1)}}$ are equal and f_1 is as smooth as is f.

Let D be the element of the enveloping algebra $\mathscr{U}(G_1)$ of G_1 guaranteed by Theorem 1.1. Let $\mathscr{E}(G_1)$ denote the algebra of differential operators generated by the first order operators d/dx_i, and choose elements $E^1,\dots,E^r$ in $\mathscr{E}(G_1)$ and polynomial functions $p^1,\dots, p^r$ on G_1 such that $D = \sum_{i=1}^r p^iE^i$ (Theorem 3.5). Now assume that f is sufficiently differentiable to carry out the following computations.

$$\begin{aligned}
\| T_f \|_{sp} &= \| T_{1_{(f_1)}} \|_{sp} \\
&\leq \int_{G_1} | Df(g_1) | \, dg_1 = \int_{G_1} \left| \sum_{i=1}^{r} p^i E^i f_1(g_1) \right| dg_1 \\
&\leq \sum_{i=1}^{r} \int_{G_1} | p^i E^i f_1(g_1) | \, dg_1 \\
&= \sum_{i=1}^{r} \int_{(-l_1)}^{(l_1)} \cdots \int_{(-l_{J+1})}^{(l_{J+1})} \int_R \cdots \int_R \left| p^i \left(\prod_{j=1}^{n+1} \exp(t_j x_j) \right) \right. \\
&\qquad \times \left. [E^i(f_1)] \left(\prod_{j=1}^{n+1} \exp(t_j x_j) \right) \right| dt_1 \cdots dt_{n+1}
\end{aligned}$$

and therefore $\| T_f \|_{sp}$ is bounded by a finite linear combination of terms of the form

$$\begin{aligned}
&\int_{(-l_1)}^{(l_1)} \cdots \int_{(-l_{J+1})}^{(l_{J+1})} \int_R \cdots \int_R \left| \prod_{j=1}^{J+1} e^{i(\pi t_j n_j / l_j)} \prod_{j=J+2}^{N+1} (t_j)^{(n_j)} \right. \\
&\qquad \times \left. \prod_{k=1}^{n+1} (d/dk_k)^{(m_k)} f_1 \left(\prod_{j=1}^{n+1} \exp(t_j x_j) \right) \right| dt_1 \cdots dt_{n+1} \\
&= \int_{(-l_1)}^{(l_1)} \cdots \int_{(-l_{J+1})}^{(l_{J+1})} \int_R \cdots \int_R \left| \prod_{j=J+2}^{n+1} (t_j)^{(n_j)} \right. \\
&\qquad \times (d/dx_1)^{(m_1)} a(\exp(t_1 x_1)) \prod_{k=2}^{n+1} (d/d(d\pi(x_k)))^{(m_k)} \\
&\qquad \times \left. f \left(\prod_{j=2}^{n+1} \exp(t_j \, d\pi(x_j)) \right) \right| dt_1 \cdots dt_{n+1} \\
&= c \int_G | p'(g) \, E' f(g) | \, dg,
\end{aligned}$$

where c is a constant, p' is a polynomial function on G, and E' is an element of the algebra $\mathscr{E}(G)$ generated by the differential operators $d/d(d\pi(x_i))$. We have shown then that $\| T_f \|_{sp}$ is bounded by a finite sum of terms of the form $\| E_p f \|_1$, where each E_p belongs to $\mathscr{E}_p(G)$. By Theorem 3.5, part (iv), each E_p must belong to $\mathscr{U}_p(G)$, and the theorem is completely proved.

Much more easily derivable directly from Theorem 1.1 is:

4.2. COROLLARY. *If f is infinitely differentiable with compact support, and if T is an irreducible multiplier representation of a connected nilpotent Lie group, then T_f is of trace class.*

4.3. EXAMPLE . Let T now be an irreducible unitary representation of a connected nilpotent Lie group. Let f be infinitely differentiable with compact support on G and let b be a Borel function of G into the circle group L for which $b(e) = 1$. Then T_{bf} is of trace class since $T_{bf} = bT_f$ and bT is an irreducible multiplier representation of G. Hence there are discontinuous functions, for instance bf, which nevertheless map to trace class operators under the representation T.

We conclude this paper with an example from more classical analysis. It is largely an example of how slippery the trace is.

Let δ be the multiplier on $R^2 \times R^2$ defined by $\delta((q_1, p_1), (q_2 p_2)) = e^{i(q_2 p_1)}$. The essentially unique irreducible δ-representation T of R^2 is defined on $L^2(R)$ by

$$(T_{(q,p)}(\varphi), \varphi) = \int_R e^{iqm} \varphi(p + m)\, \bar{\varphi}(m)\, dm.$$

If f is any integrable function on R^2, then the operator T_f is given by a kernel K_f defined by

$$K_f(m, p) = \int_R f(q, p - m)\, e^{iqm}\, dq.$$

Of course these kernels are only defined up to sets of Lebesgue measure zero in the plane.

Now since every operator of finite trace is the product of two Hilbert–Schmidt operators, we know that if f is an integrable function on R^2 for which T_f is of trace class, then the kernel $K_f(m, p) = \int_R K_1(m, n)\, K_2(n, p)\, dn$, where K_1 and K_2 are Hilbert–Schmidt kernels, i.e., square-integrable over the whole plane. Again this last equality is only up to sets of measure zero in the plane. However, if K_f is continuous in both variables, one might expect that the kernels K_1 and K_2 could be chosen so that this equality holds everywhere. However, this is not the case.

4.4. PROPOSITION. *Suppose $f(q, p) = g(q)\, h(p)$ with $h(0) \neq 0$ and h continuous, with g not continuous, with both g and h integrable. Suppose further that T_f is of trace class. (This certainly can be accomplished as in 4.3.) Then:*

(i) *The kernel K_f is continuous in both variables.*

(ii) *It is impossible to find two Hilbert–Schmidt kernels K_1 and K_2 such that $K_f(m, p) = \int_R K_1(m, n)\, K_2(n, P)\, dn$ for all m and p.*

Proof. The kernel $K_f(m, p) = \hat{g}(m)\, h(p - m)$. This is clearly continuous in both variables. Assume, by way of contradiction, that Hilbert–Schmidt kernels K_1 and K_2 do exist so that $K_f(m, p) = \int_R K_1(m, n) K_2(n, p)\, dn$ for all m and p. We would then have that

$$\int_R |K_f(m, m)|\, dm = \int_R \left| \int_R K_1(m, n) K_2(n, m)\, dn \right| dm$$

$$\leqslant \int_R \int_R |K_1(m, n) K_2(n, m)|\, dn\, dm,$$

which is finite. Hence the function $m \to K_f(m, m)$ is integrable. But this is the function $h(0)\, \hat{g}(m)$, and so $\hat{g}$ would be integrable, which implies that g would be continuous. Q.E.D.

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